Parametric Curves

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We have seen that one way to represent a curve on a plane is by writing down an equation that tells us the relation between the x and y coordinates that points on the curve need to satisfy. For example, we can use $y = f(x)$ to represent the graph of the function f, and $x^2 + y^2 = 1$ to represent the unit circle.

At the same time, we can view a given curve as the trajectory of a moving particle. If a particle moves on the plane, then its location (x, y) is a function of time t, i.e., $x = x(t)$ and $y = y(t)$. We can thus represent the trajectory of the particle by $(x, y) = (x(t), y(t))$. A curve given in this way is called a **parametric curve**. In many circumstances, the variable t is taken within some closed interval $[a, b]$. In such cases, we say the **initial point** of the curve is $(x(a), y(a))$ and the **terminal point** of the curve is $(x(b), y(b))$.

Example 1. *If $y = f(x)$ is a function, then its graph can be parameterised by $(t, f(t))$.*

Example 2. *The parametric curve $(\cos t, \sin t)$, $t \in [0, 2\pi]$ describes the unit circle with both initial and terminal points at $(1, 0)$. The curve starts at $(1, 0)$, goes in counterclockwise direction and returns to $(1, 0)$.*

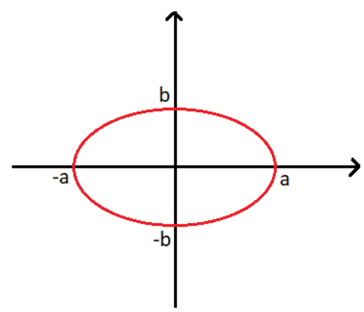
The parametric curve $(\cos t, \sin t)$, $t \in [0, \pi]$ describes the upper half of the unit circle, starting from $(1, 0)$ and ending at $(-1, 0)$.

The parametric curve $(\cos t, \sin t)$, $t \in [0, 4\pi]$ also describes the unit circle as a curve, but this time it goes around the unit circle two times. So this one is regarded as a different parametric curve compared with the previous one.

The parametric curve $(\cos(-t), \sin(-t))$, $t \in [0, 2\pi]$ also describes the unit circle as a curve, but this time it goes clockwise. So this one is also regarded as a different parametric curve compared with the previous ones.

Example 3. *The ellipse* $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ *is the trace of the parametric curve*

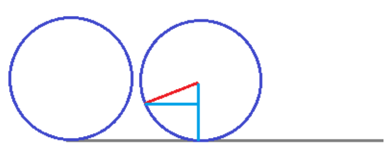
$$(a\cos t, b\sin t), 0 \leq t \leq 2\pi$$



Example 4. *The cycloid is the trace of a point on a circle as the circle rolls along a straight line. Find parametric equations for the cycloid if the circle has radius* r *and starts at origin rolling along positive* x*-axis, and the point* P *is at bottom at the beginning.*

We can let the circle travels in constant velocity r*, so its centre will be at* (rt, r) *at time* t*, and the angle formed between* P *and the point tangent to the ground is* $\frac{rt}{r} = t$*, so the parametric equation of the trace of* P *is:*

$$(rt - r\sin t, r - r\cos t) = (r(t - \sin t), r(1 - \cos t))$$



If we understand t as the time in the parametric equation $(x(t), y(t))$ for the parametric curve, then its velocity at t_0, the rate of change of the displacement at this time, is given by

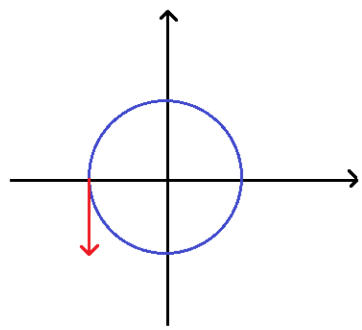
$$\lim_{t\to t_0}\left(\frac{x(t)-x(t_0)}{t-t_0}, \frac{y(t)-y(t_0)}{t-t_0}\right) = (x'(t_0), y'(t_0))$$

The vector $(x'(t_0), y'(t_0))$ is called the **tangent vector** of the parametric curve $(x(t), y(t))$ at $t = t_0$.

Example 5. *Consider the parametric curve $(\cos t, \sin t)$, $t \in [0, 2\pi]$. Find the tangent vector of the curve at the point $(-1, 0)$.*

The point $(-1, 0)$ corresponds to $t = \pi$. The tangent vector at this point is

$$((\cos t)', (\sin t)')\Big|_{t=\pi} = (-\sin t, \cos t)\Big|_{t=\pi} = (-\sin\pi, \cos\pi) = (0, -1)$$



Proposition 6. *If $(x(t), y(t))$ is a parametric curve, and $x'(t_0) \neq 0$, then the slope of the curve at $(x(t_0), y(t_0))$ is $\frac{y'(t_0)}{x'(t_0)}$.*

Proof. By the Chain Rule, $\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt}$, so $\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$.

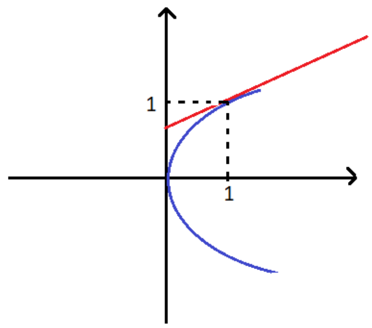
The slope is $\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{y'(t)}{x'(t)}$, so at t_0 the slope is $\frac{y'(t_0)}{x'(t_0)}$. $\square$

Example 7. *Find the line tangent to the curve $x = y^2$ at $(1, 1)$.*

The curve can be parameterised as (t^2, t). The point $(1, 1)$ corresponds to $t = 1$. The tangent vector at $t = 1$ is

$$((t^2)', t')\Big|_{t=1} = (2t, 1)\Big|_{t=1} = (2, 1)$$

This implies the tangent line of the curve passing through $(1, 1)$ is parallel to the vector $(2, 1)$, which means the slope of the tangent line is $\frac{2}{1} = 2$. So the equation of the tangent line is $y - 1 = 2(x - 1)$.



We can also compute higher order derivatives:

$$\frac{d^2y}{dx^2} = \frac{d}{dx}(\frac{dy}{dx}) = \frac{\frac{d}{dt}(\frac{dy}{dx})}{\frac{dx}{dt}} = \frac{\frac{d}{dt}(\frac{\frac{dy}{dt}}{\frac{dx}{dt}})}{\frac{dx}{dt}} = \frac{\frac{y''(t)x'(t)-y'(t)x''(t)}{[x'(t)]^2}}{x'(t)} = \frac{y''(t)x'(t) - y'(t)x''(t)}{[x'(t)]^3}$$

Example 8. *Consider the circle given by the parametric equation* $(\cos t, \sin t)$*,* $0 \le t \le 2\pi$*.*

$$\frac{d^2y}{dx^2} = \frac{y''(t)x'(t) - y'(t)x''(t)}{[x'(t)]^3} = \frac{(\sin t)''(\cos t)' - (\sin t)'(\cos t)''}{[(\cos t)']^3} = -\frac{1}{\sin^3 t}$$

When $0 < t < \pi$*,* $\frac{d^2y}{dx^2} < 0$*, the corresponding curve (upper semicircle) is concave; when* $\pi < t < 2\pi$*,* $\frac{d^2y}{dx^2} > 0$*, the corresponding curve (lower semicircle) is convex.*

The idea of tangent vector motivates the following method for computing the arc length of a parametric curve:

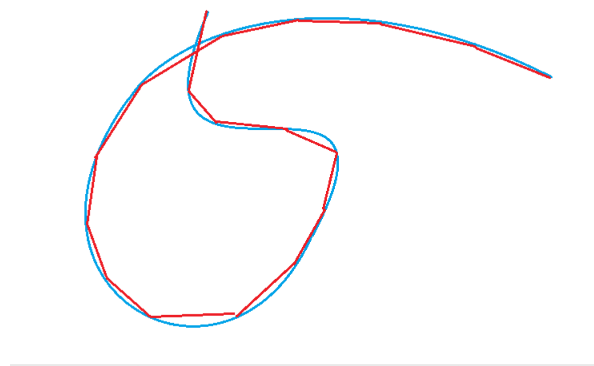
Theorem 9. *The arc length of the parametric curve* $(x(t), y(t))$*,* $t \in [a, b]$ *is*

$$\int_a^b \sqrt{[x'(t)]^2 + [y'(t)]^2}\, dt$$

Example 10. *Find the arc length of the parameterised curve* $(t - \sin t, 1 - \cos t)$ *on* $t \in [0, 2\pi]$

$$\int_0^{2\pi} \sqrt{[(t - \sin t)']^2 + [(1 - \cos t)]^2}\, dt = \int_0^{2\pi} \sqrt{2(1 - \cos t)}\, dt$$

$$= \int_0^{2\pi} \sqrt{2(2\sin^2 \frac{t}{2})}\, dt = \int_0^{2\pi} 2\sin\frac{t}{2}\, dt = 8$$



Another application of tangent vectors is to compute the area bounded by the parametric curve $(x(t), y(t))$ and the x-axis for $a \leq t \leq b$ when the curve coincide with the graph of a function $y = f(x)$:

The area is

$$\int_{x(a)}^{x(b)} f(x)\,dx = \int_a^b y(t)\,dx(t) = \int_a^b y(t)x'(t)\,dt$$

Example 11. *Find the area bounded by the parametric curve*

$$(t - \sin t, 1 - \cos t)$$

and x-axis for $0 \leq t \leq 2\pi$.

$$\int_0^{2\pi} (1 - \cos t)(t - \sin t)'\,dt = \int_0^{2\pi} (1 - \cos^2 t)^2\,dt = 3\pi$$